SPHERICAL FUNCTIONS AND INTEGRAL GEOMETRY

BY

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ABSTRACT

The paper presents two proofs of an integral geometric formula concerning n-dimensional ellipsoids. One of the proofs is based on a representation theorem for spherical functions due to Harish-Chandra.

Let E denote an n-dimensional ellipsoid centered at the origin of $\mathbb{R}^n$, and for $1 \leqq k \leqq n$, let F_k^n denote the Grassmannian manifold of k-dimensional subspaces of $\mathbb{R}^n$. The following integral-geometric formula seems to have escaped notice:

$$c_{n,k} V_n(E)^k = \int_{F_k^n} V_k(E \cap \xi)^n dm(\xi) \tag{1}$$

where $V_n(E)$ denotes the n-dimensional volume of E, $dm(\xi)$ is the normalized rotation invariant measure on F_k^n, $V_k(\xi \cap E)$ denotes the k-dimensional volume of the section $\xi \cap E$, and $c_{n,k}$ is a constant depending only on n and k. (Choosing E to be a ball we find

$$c_{n,k} = \frac{\left\{\frac{n}{2}\Gamma\left(\frac{n}{2}\right)\right\}^k}{\left\{\frac{k}{2}\Gamma\left(\frac{k}{2}\right)\right\}^n}. \tag{2}$$
)

For $k = 1$ this formula represents simply the rule for integration in polar coordinates and is valid for any symmetric star-shaped body in $\mathbb{R}^n$. But for $k > 1$, (1) does not appear to reduce to any well-known formula, and we do not know in what generality such a formula is valid.

In §6 we shall see that a number of other integration formulas can be obtained

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together with (1), and that these are all consequences of a representation theorem for spherical functions developed by Harish-Chandra. In the first sections we shall give an independent, "elementary" derivation of (1).

1. Homogeneous spaces and multiplier functions

Let G be a topological group and M a topological G-space, i.e., we assume defined a continuous map $G \times M \to M$ denoted by $(g,x) \to gx$, and satisfying $ex = x$ and $(g'g'')x = g'(g''x)$. We denote by $Z(G,M)$ the group of functions (cocycles, multiplier functions) from $G \times M$ to the positive reals satisfying

$$\sigma(g'g'',x) = \sigma(g',g''x)\sigma(g'',x). \tag{3}$$

The subgroup of multiplier functions (m.f.'s) having the form

$$\sigma(g,x) = f(gx)/f(x) \tag{4}$$

where f is a continuous function from M to the positive reals, will be denoted $B(G,M)$. The quotient group will be denoted $H(G,M)$. If K is a subgroup of G, we denote by $Z_K(G,M)$ the subgroup of m.f.'s satisfying $\sigma(k,x) = 1$ for $k \in K$, $x \in M$.

LEMMA 1. *If K is a compact subgroup of G which is transitive on M, then the natural map of $Z_K(G,M)$ into $H(G,M)$ is an isomorphism onto.*

PROOF. If $\sigma \in Z_K(G,M) \cap B(G,M)$, then $\sigma(g,x) = f(gx)/f(x)$ where $f(kx) = f(x)$ for all k,x. But then f is a constant and so $\sigma = 1$. This shows that the map in question is one-one. On the other hand, if $\sigma \in Z(G,M)$, form

$$\sigma'(g,x) = \frac{\int_k \sigma(k,gx)dk}{\int_K \sigma(k,x)\,dk}\sigma(g,x) = \frac{\int_k \sigma(kg,x)\,dk}{\int_K \sigma(k,x)\,dk} .$$

Here dk denotes Haar measure on K. Then σ and σ' are congruent modulo $B(G,M)$ and $\sigma' \in Z_K(G,M)$. It follows that the image of $Z_K(G,M)$ is all of $H(G,M)$.

Now let $M = G/H$ for some closed subgroup H of G. Let x_0 denote the coset H in G/H. If $\sigma \in Z(G,M)$, then $\sigma(h,x_0)$ for $h \in H$ defines a mutiplicative homomorphism from H to the positive reals. We call such a homomorphism a character. Let $\chi_\sigma(h) = \sigma(h,x_0)$.

LEMMA 2. *If K is transitive on G/H, then each $\sigma \in Z_K(G,G/H)$ is determined by the character χ_σ.*

PROOF. Assume that $\sigma(h,x_0) = 1$ for all $h \in H$. If $g \in G$ then $g = kh$, with

$k \in K$, $h \in H$, so that $\sigma(g, x_0) = \sigma(kh, x_0) = \sigma(k, hx_0)\sigma(h, x_0) = 1$. Finally $\sigma(g, g'x_0) = \sigma(gg', x_0)/\sigma(g', x_0) = 1$ so that $\sigma \equiv 1$.

Combining the above lemmas we see that if K is compact and transitive on G/H, then the cohomology group $H(G, G/H)$ is isomorphic to a subgroup of the positive character group of H.

2. Multiplier functions on the Grassmannian

To illustrate the foregoing let $G = SL(n, \mathbb{R})$, the group of $n \times n$ unimodular matrices, and let $M = F_k^n$, the Grassmannian variety of k-planes through the origin in $\mathbb{R}^n$. If $K = SO(n)$, then K is compact and transitive on M. Moreover $M = G/H$ where H can be taken as the subgroup of matrices

$$H = \left\{\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}\right\}$$

where A is a $k \times k$ matrix and D is an $n - k \times n - k$ matrix. It is not hard to show that the commutator group of H consists of those matrices in H with $\det A = \det D = 1$. Since a character on H takes the value 1 on the commutator, we see that the character group of H is one-dimensional and consists of powers of the character ω defined by

$$\omega \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det A.$$

Now it is not hard to determine the multiplier function in Z_K which corresponds to the character ω. Namely, let $g \in SL(n, \mathbb{R})$ and $x \in F_k^n$ so that g is a linear transformation of $\mathbb{R}^n$ and x is a subspace of $\mathbb{R}^n$. Assuming a fixed Euclidean (i.e., Hilbert space) structure on $\mathbb{R}^n$, each subspace is endowed with a Euclidean structure, and we can talk of k-dimensional volumes of subsets of x. The transformation g induces a linear transformation of x onto gx, and the expression

$$\sigma_k(g, x) = \frac{V_k(g\Delta)}{V_k(\Delta)} \tag{5}$$

where V_k denotes k-dimensional volume, is independent of the subset Δ of x that is chosen. One sees readily that σ_k is a multiplier function in Z_K. Moreover, if $h \in H$, $h = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$, and $x = x_0$ then $h|x_0$ is represented by the matrix A. Then clearly $\sigma(h, x_0) = \det A$. We have thus proved

LEMMA 3. *The group $H(SL(n,\mathbb{R}),F_k^n)$ is one-dimensional and its elements are represented by the multiplier functions*

$$\sigma(g,x) = \sigma_k^t(g,x)$$

where t is real and σ_k is defined by (5).

3. Radon-Nikodym derivatives as multiplier functions

Suppose, more generally, that G is a Lie group, $M = G/H$ a homogeneous space and K a compact subgroup of G that is transitive on M. M has a C^∞ manifold structure and G acts on M by C^∞ diffeomorphisms. A measure on a manifold will be called "smooth" if it is expressed in local coordinates by $\psi(x_1,\cdots,x_l)$ $dx_1\cdots dx_2$ where ψ is a C^∞ function. It will be called "strictly positive" if ψ is strictly positive. If μ is a measure on M, we denote by $g\mu$ the measure defined by $g\mu(A) = \mu(g^{-1}A)$. Inasmuch as K is transitive on M, there will exist a unique probability measure on M, denoted m_M satisfying $km_M = m_M$ for all $k \in K$. Now let μ be any smooth probability measure on M. Then for each $g \in G$, the measure $g\mu$ is smooth. The measure defined by

$$\bar{\mu}(A) = \int_K k\mu(A)dk$$

will again be a smooth measure on M. Since $\bar{\mu}$ is K-invariant, it follows that $\bar{\mu} = m_M$ and therefore we conclude that m_M is a smooth measure. Similarly, by choosing μ strictly positive, we may conclude that m_M is strictly positive.

Now let μ_1 and μ_2 be two strictly positive smooth measures on M. Then they are each absolutely continuous with respect to the other and we may form the Radon-Nikodym derivative $d\mu_1/d\mu_2$. This is a function defined almost everywhere on M. However, in local coordinates, this will be the ratio of two nonvanishing C^∞ functions, and hence there is a unique continuous version of this derivative.

LEMMA 4. *Let*

$$\sigma_M(g,x) = \frac{dg^{-1}m_M}{dm_M}(x).$$

Then σ_M is a multiplier function belonging to $Z_K(G,M)$.

PROOF. We make use of the following two rules for Radon-Nikodym derivatives:

(i) $$\frac{d\mu_1}{d\mu_3} = \frac{d\mu_1}{d\mu_2}\cdot\frac{d\mu_2}{d\mu_3}$$

(ii)
$$\frac{dg\mu_1}{dg\mu_2}(x) = \frac{d\mu_1}{d\mu_2}(g^{-1}x)$$

valid almost everywhere, assuming the measures μ_i are in the same absolute continuity class. When the μ_i are strictly positive smooth measures on M, then these equalities are valid everywhere. Hence

$$\begin{aligned}\sigma_M(g_1g_2,x) &= \frac{dg_2^{-1}g_1^{-1}m_M}{dm_M}(x) = \frac{dg_2^{-1}g_1^{-1}m_M}{dg^{-1}m_M}(x)\,\frac{dg_2^{-1}m_M}{dm_M}(x)\\ &= \frac{dg_1^{-1}m_M}{dm_M}(g_2x)\frac{dg_2^{-1}m_M}{dm_M}(x) = \sigma_M(g_1,g_2x)\sigma_M(g_2,x).\end{aligned}$$

Finally, since m_M is K-invariant, it follows that $\sigma_M \in Z_K(G,M)$.

4. Proof of the integration formula (1)

Let us return to the example $G = SL(n,\mathbb{R})$, $M = F_k^n$, $K = SO(n)$. We have two explicit examples of multiplier functions in $Z_K(G,M)$, σ_k and σ_M. Since, by Lemma 3, this group of m.f.'s is one-dimensional, we must have $\sigma_M = \sigma_k^t$ for some real number t. In fact, we have

LEMMA 5. *The multiplier functions σ_k and $\sigma_{F_k^n}$ are related by*

(6)
$$\sigma_{F_k^n}(g,x) = \sigma_k^{-n}(g,x).$$

PROOF. From the foregoing discussion we know that a relationship of the form $\sigma_{F_k^n} = \sigma_k^{\alpha_{n,k}}$ is valid. To show that $\alpha_{n,k} = -n$, it suffices to check (6) for any g,x for which $\sigma_k(g,x) \neq 1$. We choose

$$g = h_a = \begin{bmatrix} a & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & \dots & \cdot \\ & \cdot & \cdot & & \\ & & & 1 & 0 \\ 0 & 0 & \dots & 0 & a^{-1} \end{bmatrix} \qquad a \neq 1,$$

and $x = x_0$. Then $\sigma_k(h_a,x_0) = a$. Now let μ be any smooth, strictly positive measure on M. Then

$$\frac{dg^{-1}\mu}{d\mu}(x) = \frac{dg^{-1}\mu}{dg^{-1}m_M}(x)\,\frac{dg^{-1}m_M}{dm_M}(x)\,\frac{dm_M}{d\mu}(x) = \frac{f(gx)}{f(x)}\,\sigma_M(g,x)$$

where $f(x) = d\mu/dm_M(x)$. If we set $\sigma'(g,x) = dg^{-1}\,\mu/d\mu(x)$, then since $h_ax_0 = x_0$ we will have

$$\sigma_M(h_a,x_0) = \sigma'(h_a,x_0)$$

As a result, to compute $\sigma_M(h_a, x_0)$, we can take any convenient measure defined in a local coordinate system about x_0, and form the corresponding Radon-Nikodym derivative. Since in ordinary Euclidean space a diffeomorphism transforms Euclidean measure by multiplying it by the jacobian of the transformation, we see that our problem reduces to computing the jacobian of the transformation h_a in an appropriate coordinate system on F^n_k. Such a coordinate system is obtained by assigning to the coset gH the matrix $X(g) = CA^{-1}$ where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. One checks that this depends only on the coset gH, and it provides a coordinatization of the neighborhood of x_0 for which A is invertible. Moreover x_0 corresponds to the 0 matrix. Now write

$$h_a = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$$

where P is a $k \times k$ matrix and Q is an $n-k \times n-k$ matrix. Then

$$h_a \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} PA & PB \\ QC & QD \end{pmatrix}$$

so that $X(h_a g) = QCA^{-1}P^{-1} = QX(g)\,P^{-1}$. In this coordinate system the transformation h_a is linear and its jacobian is readily computed to be a^n. This completes the proof of the lemma.

We are now in a position to prove the identity (1).

THEOREM 1. *If E is an n-dimensional ellipsoid with center at the origin of $\mathbb{R}^n$ then*

$$\text{(1 bis)} \qquad c_{n,k}V_n(E)^k = \int_M V_k(\xi \cap E)^n dm_M(\xi)$$

where $M = F^n_k$ is the Grassmannian variety of k-dimensional subspace of $\mathbb{R}^n$, and m_M is the rotation invariant probability measure on M.

PROOF. We note that if the equality in question is valid for an ellipsoid E it is also valid for every dilation of E. Hence we may assume that E has the same volume as the n-dimensional ball B_n. Then $E = g^{-1}B_n$ for some $g \in SL(n, \mathbb{R})$. It follows that

$$V_k(\xi \cap E) = \frac{V(\xi \cap E)}{V_k(g(\xi \cap E))} V_k(g(\xi \cap E)) = \sigma_k^{-1}(g, \xi) V_k(g\xi \cap B_n)$$

Now $g\xi \cap B_n$ is a k-dimensional section of B_n and its volume is that of the k-dimensional unit ball $V_k(B_k)$ $\left(= \dfrac{\pi^{k/2}}{\frac{k}{2}\Gamma\left(\frac{k}{2}\right)} \right)$. Hence

$$\int_M V_k(\xi \cap E)^n dm_M(\xi) = V_k(B_k)^n \int_M \sigma_k^{-n}(g,\xi) dm_M(\xi) = V_k(B_k)^n \int_M \sigma_M(g,\xi) dm_M(\xi)$$

$$= V_k(B_k)^n \int_M \frac{dg^{-1}m_M}{dm_M}(\xi)\, dm_M(\xi) = V_k(B_k)^n \cdot m_M(gM) = \frac{V_k(B_k)^n}{V_n(B_n)^k} V_n(E)^k.$$

This proves the theorem.

5. Unitary multiplier functions and flag manifolds

All the Grassmannian varieties F_k^n occurring in the preceding sections are equivariant images of a single homogeneous manifold F^n. We say that a G-space N is an equivariant image of a G-space M if there exists a map ϕ of M onto N satisfying $\phi(gx) = g\phi(x)$ for $x \in M$. Then if we take F^n to be the space of all "flags" in $\mathbb{R}^n$, i.e., of all $(n-1)$-tuples of subspaces $\xi_1 \subset \xi_2 \subset \cdots \subset \xi_{n-1} \subset \mathbb{R}^n$, where $\dim \xi_i = i$, F^n will be a G-space for $G = SL(n,\mathbb{R})$, and the natural map of F^n to F_k^n is an equivariant map. The orthogonal group K is transitive on F^n and the cohomology group $H(G,F^n)$ is isomorphic to $Z_K(G,F^n)$. We proceed to determine the group $Z_K(G,F^n)$.

Let ξ_0 be the flag in F^n whose ith component is the subspace of all vectors in $\mathbb{R}^n$ whose last $n-i$ components vanish. Then the isotropy group of ξ_0 consists of the subgroup H of upper triangular matrices $\{h = (a_{ij}) \mid a_{ij} = 0,\ i > j\}$ and we have $F^n = G/H$. The group $Z_K(G,F)$ is isomorphic to a subgroup of the positive character group of H and the latter is quite easy to determine. One sees that such characters have the form

$$x(h) = a_{11}^{t_1} a_{22}^{t_2} \cdots a_{n-1,n-1}^{t_{n-1}}$$

Now we can lift the m.f. σ_k from $G \times F_k^n$ to $G \times F^n$ by identifying $\sigma_k(g,(\xi_1,\cdots,\xi_{n-1}))$ with $\sigma_k(g,\xi_k)$. The resulting m.f. belongs to $Z_K(G,F)$ and we have

$$\chi_{\sigma_k}(h) = \sigma_k(h,\xi_0) = a_{11}a_{22}\cdots a_{kk}$$

From this it follows that the σ_k generate all of $Z_K(G,F^n)$ and that any m.f. in $Z_K(G,F^n)$ can be written

$$\sigma = \sigma_1^{r_1}\sigma_{2n}^{r_2} \cdots \sigma_{n-1}^{r_{n-1}}. \tag{7}$$

Now suppose that a m.f. $\sigma \in Z_K(G,F)$ has the property that for *all* $g \in G$

$$\int_{F^n} \sigma(g,\xi)dm_{F^n}(\xi) = 1, \tag{8}$$

where, as usual, m_{F^n} denotes the K-invariant probability measure on F^n. Call such a m.f. *unitary*. We shall show that (8) implies an integration formula similar to (1). In fact, let E be an n-dimensional ellipsoid centered at the origin in $\mathbb{R}^n$ with $V_n(E) = V_n(B_n)$. Then there exists $g \in SL(n,\mathbb{R})$ with $E = g^{-1}B_n$. Suppose σ has the form (7). We may write

$$\sigma_k(g,\xi) = \frac{V_k(g(\xi_k \cap E))}{V_k(\xi_k \cap E)} = \frac{V_k(g\xi_k \cap B_n)}{V_k(\xi_k \cap E)} = \frac{V_k(B_k)}{V_k(\xi_k \cap E)}.$$

Hence

$$\int_{F^n} \left(\frac{V_1(\xi_1 \cap E)}{V_1(B_1)}\right)^{-r_1} \cdots \left(\frac{V_{n-1}(\xi_{n-1} \cap E)}{V_{n-1}(B_{n-1})}\right)^{-r_{n-1}} dm_{F^n}(\xi) = 1$$

or

$$\int_{F^n} V_1(\xi_1 \cap E)^{-r_1} \cdots V_{n-1}(\xi_{n-1} \cap E)^{-r_{n-1}} dm_{F^n}(\xi) = c_{r_1 \cdots r_{n-1}}$$

Replacing E by λE we have $\lambda^n = V_n(E)$ and

$$\int_{F^n} V_1(\xi_1 \cap E)^{-r_1} \cdots V_{n-1}(\xi_{n-1} \cap E)^{-r_{n-1}} dm_{F^n}(\xi) = c_{r_1 \cdots, r_{n-1}} V_n(E)^{-\Sigma_i r_i/n}$$

We thus find

LEMMA 6. *If $\sigma_1^{r_1}\sigma_2^{r_2}\cdots\sigma_{n-1}^{r_{n-1}}$ is a unitary m.f. in $Z_K(G,F^n)$, then*

$$\int_{F^n} V_1(\xi_1 \cap E)^{-r_1} \cdots V_{n-1}(\xi_{n-1} \cap E)^{-r_{n-1}} dm_{F^n}(\xi) = c_{r_1 \cdots, r_n} V_n(E)^{-1/n\Sigma_i r_i} \tag{9}$$

for any ellipsoid E centered at the origin in $\mathbb{R}^n$.

Theorem 1 now follows from Lemma 6 and the fact that $\sigma_M = \sigma_K^{-n}$ is unitary for each k. This latter fact follows from Lemma 5 and by generalizing the setup in Lemma 5 we may obtain wide family of unitary multiplier functions. Namely, let M be any equivariant image of F, $\phi: F^n \to M$. Set

$$\sigma_M(g,\xi) = \frac{dg^{-1}m_M}{dm_M}(\phi(\xi))$$

LEMMA 7. *$\sigma_M \in Z_K(G,F^n)$ and is unitary.*

The proof of the fiirst assertion is identical to that of Lemma 4 except that we make use of the equivariance of ϕ so that

$$\frac{dg_2^{-1}g_1^{-1}m_M}{dg_2^{-1}m_M}(\phi(\xi)) = \frac{dg_1^{-1}m_M}{dm_M}(g_2\phi(\xi)) = \frac{dg_1^{-1}m_M}{dm_M}(\phi(g_2\xi)) = \sigma_M(g_1, g_2\xi).$$

σ_M is unitary because m_M is the image under ϕ of m_{F^n} so that

$$\int_{F^n} \frac{dg^{-1}m_M}{dm_M}(\phi(\xi))\, dm_{F^n}(\xi) = \int_M \frac{dg^{-1}m_M}{dm_M}(\xi)\, dm_M(\xi) = \int_M dg^{-1}m_M(\xi) = 1.$$

Now we have already noted that F_k^n is an equivariant image of F^n. But we can also define the flag manifold $F^n_{i_1,\cdots i_r}$ of partial flags $\xi = (\xi_{i_1}, \cdots, \xi_{i_r})$ where $\xi_{i_1} \subset \cdots \subset \xi_{i_r}$ are subspaces of the dimensions indicated. Clearly for each subset $i_1 < i_2 < \cdots < i_r$ of $(1, 2, \cdots n-1)$ we obtain an equivariant image of F^n and we can form the unitary m.f. $\sigma_{F^n_{i_1,\cdots,i_r}}$. A computation similar to that of Lemma 5 leads to the identification

LEMMA 8.

$$\sigma_{F^n_{i_1\cdots i_r}} = \sigma_{i_1}^{-i_2}\sigma_{i_2}^{i_1-i_3}\cdots\sigma_{i_r}^{i_{r-1}-n}.$$

This leads to the following generalization of Theorem 1:

THEOREM 2. *If E is an ellipsoid centered at the origin of $\mathbb{R}^n$ then*

$$(10)\quad \int V_{i_1}(\xi_{i_1} \cap E)^{i_2} V_{i_2}(\xi_{i_2} \cap E)^{i_3 - i_1} \cdots V_{i_r}(\xi_{i_r} \cap E)^{n - i_{r-1}} dm(\xi) = c(i_1, \cdots, i_r) V_n(E)^{i_r}$$

where the integration is taken over the flag manifold $F^n_{i_2,\cdots,i_r}$, and the constant $c(i_1, \cdots, i_r)$ is chosen so that the result is valid for $E = B_n$.

REMARK. Actually, (10) can also be obtained as a direct consequence of (1) by applying the latter repeatedly and observing that $F^n_{i_1,\cdots i_r}$ is a fiber bundle over $F^n_{i_2\cdots i_r}$ with fiber $F^{i_2}_{i_1}$.

We now turn to the question of determining *all* the unitary m.f. in $Z_K(G, F^n)$. This will determine all the integration formulas of the form (9). Another instance of a formula of this type occurs in ([1], lemma 8.3).

6. Multiplier functions and spherical functions

Let G be a semi-simple Lie group with finite center, K a maximal compact subgroup, M a G-space on which K acts transitively. One then has

LEMMA 9. *If $\sigma \in Z_K(G, M)$ and*

$$\phi(g) = \int_M \sigma(g, x)\, dm_M(x)$$

then

$$(12)\qquad \int_K \phi(g_1 k g_2)\, dk = \phi(g_1)\,\phi(g_2).$$

PROOF. Straightforward verification.

A function satisfying (12) is called a *spherical function*. Note that a spherical function satisfies $\phi(gk) = \phi(g) = \phi(kg)$ for $k \in K$. We refer the reader to ([3], chap. X) for a comprehensive treatment of the theory of spherical functions. One of the principal results of this theory is the theorem of Harish-Chandra (th. 6.16) which gives an explicit representation of any spherical function. This theorem implies that a converse of Lemma 9 is valid, and that any function satisfying (12) arises by means of the representation (11) for an appropriate space M. We shall give the details for the special case of interest to us, $G = SL(n, R)$.

Let $g \in SL(n,R)$. Then g has a unique decomposition $g = k(\exp a)n$ where $k \in SO(n)$, a is a diagonal matrix, and n is an upper triangular matrix with 1's along the diagonal. We set $A(g) = a$. The set of all matrices of the form $A(g)$, i.e. the set of all diagonal matrices with trace 0, forms a vector space whose dual we shall denote by Λ. We denote by W (for Weyl group) the symmetric group on n elements. W acts on the space of diagonal matrices by permuting diagonal elements, and so it also acts on the dual space Λ. We shall write λ^ω for the transform of the element λ by the permutation ω. Finally let ρ denote the element of λ^ω defined by

$$\rho \begin{bmatrix} a_1 & 0 & \dots & 0 \\ 0 & a_2 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & & & a_n \end{bmatrix} = a_1 + 2a_2 + \cdots + na_n = -\{(n-1)a_1 + (n-2)a_2 + \cdots + a_{n-1}\}$$

THEOREM 3. *(Harish-Chandra) Every spherical function on $SL(n, \mathbb{R})$ has the representation*

$$\phi_\lambda(g) = \int_K e^{\lambda(A(gk))}\, dk \tag{13}$$

where $K = SO(n)$, and $\lambda \in \Lambda$. Moreover $\phi_\lambda = \phi_{\lambda'}$ if and only if $\lambda' = \lambda + \rho - \rho^\omega$ for some $\omega \in W$.

We shall now see that the representation (13) can be put in the form (11) with M the flag manifold F^n. We need

LEMMA 10. *Let H denote the subgroup of upper triangular matrices in $SL(n,R)$ and set*

$$R(g, kH) = A(gk).$$

Then R is well defined for $g \in G$, $k \in K$ and satisfies

(14) $$R(g_1g_2, kH) = R(g_1, g_2kH) + R(g_2, kH).$$

PROOF. To show that R is well defined we must show that if $k_1H = k_2H$ then $A(gk_1) = A(gk_2)$. But if $k_2^{-1}k_1 \in H \cap K$, then $k_2^{-1}k_1$ is a diagonal matrix. Writing $gk_2 = ken$, with $e = \exp A(gk_2)$, then $gk_1 = kenk_2^{-1}k_1 = kek_2^{-1}k_1n' = kk_2^{-1}k_1en'$, whence $A(gk_1) = A(gk_2)$.

To prove (14) we decompose g_2k in the form $k'en$ so that $g_1g_2k = g_1k'en$. Let $g_1k' = k''e'n'$; then $g_1g_2k = k''e'n'en$. Since e is a diagonal matrix and n' is upper triangular with 1's on the diagonal, $n'e$ will have the same diagonal as e, and therefore $n'e$ can be rewritten en'' where n'' is upper triangular with 1's on the diagonal. Hence one obtains

$$A(g_1g_2k) = A(g_1k') + A(g_2k)$$

so that

$$R(g_1g_2, kH) = R(g_1, k'H) + R(g_2, kH).$$

Since $\exp A(g_2, k) \cdot n \in H$, $k'H = g_2kH$ and the desired result follows.

From the foregoing lemma it follows that the integrand in (13) can be written $\sigma_\lambda(g, \xi)$, $\xi \in F^n$, where σ_λ is a m.f. (not necessarily taking positive values). If λ is real valued then $\sigma_\lambda \in Z_K(G, F^n)$ and λ is determined from σ_λ by $\exp_\lambda(a) = \sigma_\lambda(\exp(a), \xi_0)$.

Now observe that $\phi(g) \equiv 1$ is a spherical function. It evidently corresponds to $\lambda = 0$ and by the Harish-Chandra theorem it can be expressed in the form (13) exactly for those σ_λ, λ of the form $\rho - \rho^\omega$. But this means that σ is unitary iff $\sigma = \sigma_\lambda$ for $\lambda = \rho - \rho^\omega$. Thus

THEOREM 4. *The formula*

$$\int_{F^n} \sigma(g, \xi)\, dm_{F^n}(\xi) = 1$$

is valid for all $g \in G$ if and only if $\sigma = \sigma_\lambda$ with $\lambda = \rho - \rho^\omega$ for some permutation ω.

It remains to compute these σ_λ in terms of the basis $\sigma_1, \cdots, \sigma_{n-1}$ of $Z_K(G, F^n)$. Let $\omega \in W$; then $\omega(1), \omega(2), \cdots \omega(n)$ is a permutation of the first n numbers. Then

$$\rho^\omega \begin{bmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \cdot\cdot & \cdots\cdots & & \\ 0 & 0 & \dots & \alpha_n \end{bmatrix} = \rho \begin{bmatrix} \alpha_{\omega(1)} & 0 & \dots & 0 \\ 0 & \alpha_{\omega(2)} & \dots & 0 \\ \cdots & \cdots\cdots & & \\ 0 & 0 & \dots & \alpha_{\omega(n)} \end{bmatrix} = \alpha_{\omega(1)} + \cdots + n\alpha_{\omega(n)}$$

Let d denote the diagonal matrix with entries $\exp \alpha_i$ on the diagonal. σ_{ρ^ω} is determined by its values for all (d, ξ_0) with d as above. Now

$$\exp \alpha_j = \sigma_j(d, \xi_0)/\sigma_{j-1}(d_1\xi_0) \qquad (\sigma_0 \equiv 1, \sigma_n \equiv 1)$$

so that

$$\sigma_\rho(d_1\xi_0) = \exp \sum_{i=1}^{n} i\alpha_{\omega(i)} = \prod_{i=1}^{n} \sigma_{\omega(i)}(d_1\xi_0)^i/\sigma_{\omega(i)-1}(d_1\xi_0)$$

and

$$\sigma_\rho = \prod_{i=1}^{n} \sigma^i_{\omega(i)}\sigma^{-i}_{\omega(i)-1} = \prod_{i=1}^{n} \sigma_j^{\omega^{-1}(j)}\sigma_{j-1}^{-\omega^{-1}(j)} = \prod_{j=1}^{n-1} \sigma_j^{\omega^{-1}(j)-\omega^{-1}(j+1)}.$$

When ω is the identity permutation we obtain $\sigma_\sigma = \prod_{j=1}^{n-1}\sigma_j^{-1}$. Replacing ω by ω^{-1} we finally deduce

THEOREM 5. *A multiplier function is unitary if and only if it has the form*

$$\sigma = \prod \sigma_j^{\omega(j+1)-\omega(j)-1}$$

for some permutation ω.

Applying Lemma 6 we find

THEOREM 6. *For every permutation ω of the integers $(1,2,\cdots,n)$, there exists a constant c_ω such that if E is an n-dimensional ellipsoid in $\mathbb{R}^n$ centered at the origin, then*

$$\int_{F^n} \prod_{j=1}^{n-1} V_j(\xi_j \cap E)^{\omega(j)-\omega(j+1)+1} dm_{F^n}(\xi) = c_\omega V_n(E)^{n-\omega(n)} .$$

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